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Falsity of Wang's conjecture on stars

C.S. Karuppan Chetty [a,*], S. Maria Arulraj [b]

[a] Department of Mathematics, Regional Engineering College, Tiruchirappalli-620 015, India

[b] Department of Mathematics, St. Joseph's College, Tiruchirappalli-620 002, India

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Abstract

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. E.T.H. Wang called a matrix $B \in \Omega_n$ a star if $\operatorname{per}(\alpha B + (1-\alpha)A) \leqslant \alpha \operatorname{per}(B) + (1-\alpha)\operatorname{per}(A)$ for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$ and conjectured in 1979 that for $n \geqslant 3$, permutation matrices are the only stars. In this paper we disprove Wang's conjecture for $n = 3$, by showing that PBQ is a star where

$$B = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} \oplus 1, \quad 0 \leqslant x \leqslant 1$$

and P and Q are permutation matrices. We also establish that the only stars in Ω_3 are PBQ as defined above.

Keywords: Permanent; Star; Mate

1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. An interesting area in the study of permanents is to inquire whether the permanent function is convex on Ω_n i.e., to see the validity of the inequality

$$\operatorname{per}(\alpha B + (1-\alpha)A) \leqslant \alpha \operatorname{per}(B) + (1-\alpha)\operatorname{per}(A) \tag{1.1}$$

* Corresponding author.

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for all $A, B \in \Omega_n$ and for all $\alpha \in [0, 1]$. Though the result is true for $n = 2$, it is not so for $n \geqslant 3$ and it was shown by a counterexample given by Marcus and quoted by Perfect [1]. In view of the falsity of the convexity of the permanent function restricting B to some particular matrices in Ω_n, the validity of (1.1) for all $A \in \Omega_n$ and for all $\alpha \in [0, 1]$ was investigated by many authors. In this direction, Perfect [1] proved that

$$\operatorname{per}\left(\frac{I + A}{2}\right) \leqslant \frac{1 + \operatorname{per}(A)}{2}$$

for all $A \in \Omega_n$, where $I = I_n$, the $n \times n$ identity matrix. Subsequently, Brualdi and Newman [2] generalized the above result and proved that

$$\operatorname{per}(\alpha I_n + (1 - \alpha)A) \leqslant \alpha + (1 - \alpha) \operatorname{per}(A) \tag{1.2}$$

for all $A \in \Omega_n$ and for all $\alpha \in [0, 1]$. Moreover, they established that (1.1) will hold for all $\alpha \in [0, 1]$ and for all $A \in \Omega_n$, iff for all $A \in \Omega_n$

$$\sum_{i,j} b_{ij} \operatorname{per}(A_{ij}) \leqslant \operatorname{per}(B) + (n - 1) \operatorname{per}(A), \tag{1.3}$$

where $B = (b_{ij})$ and A_{ij} is the $(n - 1) \times (n - 1)$ matrix obtained by deleting the ith row and jth column of A and equality in (1.3) holds iff $A = B$. Wang [3] called such a matrix B satisfying (1.1) for all $A \in \Omega_n$ and for all $\alpha \in [0, 1]$ as a star. Wang [3] proved that: (i) every 2×2 doubly stochastic matrix is a star and (ii) if $B \in \Omega_n$ is a star then $\operatorname{per}(B) \geqslant 1/2^{n-1}$. Using the definition of star and elementary properties of permanent function one can easily prove the following.

Lemma 1.1. *If $B \in \Omega_n$, the following are equivalent*

(i) B is a star

(ii) B^{T}, the transpose of B is a star

(iii) PBQ is a star for any two permutation matrices P and Q.

Applying the above lemma it follows from the inequality (1.2) that any permutation matrix is a star. Wang believed that for all $n \geqslant 3$, the only stars are permutation matrices and hence proposed the following conjecture.

Conjecture 1.1. For $n \geqslant 3$, $B \in \Omega_n$ is a star iff B is a permutation matrix.

In this paper we disprove this conjecture for $n = 3$ by showing that

$$B = \begin{bmatrix} x & 1 - x \\ 1 - x & x \end{bmatrix} \oplus 1, \quad 0 \leqslant x \leqslant 1$$

is a star. We also establish that the only stars in Ω_3 are PBQ where B is as defined above and P and Q are permutation matrices.

Throughout this paper for brevity let us use the notation $M(a,b;c,d)$ to denote the 3×3 doubly stochastic matrix

$$\begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{bmatrix}.$$

Following essentially the same notation used by Minc [4] for integers r, $n(1\leqslant r\leqslant n)$, let $Q_{r,n}$ denote the set of all sequences $(i_1,\dots,i_r)$ such that $1\leqslant i_1<\cdots<i_r\leqslant n$. For fixed $\alpha,\beta\in Q_{r,n}$ let $A(\alpha|\beta)$ be the submatrix of A obtained by deleting the rows α and columns β of A, and let $A[\alpha|\beta]$ denote the submatrix of A formed by the rows α and columns β of A. We also use the following result (Theorem 1.4, Minc [4]).

Theorem 1.1. *Let $A=(a_{ij})$ and $B=(b_{ij})$ be $n\times n$ matrices. Then*

$$\operatorname{per}(A+B)=\sum_{r=0}^{n}S_r(A,B), \tag{1.4}$$

where

$$S_r(A,B)=\sum_{\alpha,\beta\in Q_{r,n}}\operatorname{per}(A[\alpha|\beta])\operatorname{per}(B(\alpha|\beta))$$

and per $(A[\alpha|\beta])=1$ when $r=0$ and per $(B(\alpha|\beta))=1$ when $r=n$.

2. Counterexample to Wang's conjecture

In this section we disprove Wang's conjecture by showing that $M(x,1-x;1-x,x)$, $0\leqslant x\leqslant 1$ is a star.

Theorem 2.1. *$B=M(x,1-x;1-x,x)$ is a star for $0\leqslant x\leqslant 1$.*

Proof. The proof of this theorem is rather simple and basically it is a tedious exercise in elementary calculus. So we omit the details of the proof and we simply summarize it in a few lines.

Taking $A=M(a,b;c,d)$, for B to be a star, in view of (1.3), it is sufficient to verify that

$$\begin{aligned} F(a,b;c,d) &= \operatorname{per}(B)+2\operatorname{per}(A)-\sum_{i,j=1}^{3}b_{ij}\operatorname{per}(A_{ij}) \\ &= 2x^2-x[2+(b+c)-(a+d)+2(a+d)^2 \\ &\quad -2(b+c)^2-4ad+4bc]-1+4(a+d)+5(b+c) \\ &\quad -6(a+d)(b+c)-2(b+c)^2-13ad \\ &\quad -9bc+4ad(a+d+2b+2c) \\ &\quad +4bc(2a+2d+b+c)\geqslant 0. \end{aligned} \tag{2.1}$$

Using the necessary conditions for F to have a local minimum, we first prove that $F(a,b;c,d) \geqslant 0$ for all A in the interior of Ω_3. We complete the proof of the theorem, by proving that $F(a,b;c,d) \geqslant 0$ for all A on the boundary of Ω_3 by considering the four possible cases
(i) $a=b=c=0$
(ii) $a=b=0$
(iii) $a=d=0$
(iv) $a=0$

As a consequence we have the following corollary.

Corollary 2.1. *A matrix of the form PBQ where B is as in Theorem 2.1 and P and Q are* 3×3 *permutation matrices is a star.*

3. Characterization of 3×3 stars

The purpose of this section is to establish that the only star in Ω_3 is $M(x,1-x;1-x,x)$ where $0\leqslant x\leqslant 1$ upto permutations of rows and columns.

Any matrix B in Ω_3 can be classified in one and only one of the following four categories.

(i) B has no zero entry.
(ii) B has precisely one or two or three zeros.
(iii) B has precisely four zeros so that $B = PXQ$ where P and Q are some permutation matrices and

$$X = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} \oplus 1, \quad 0 < x < 1.$$

(iv) B has precisely six zeros so that B is a permutation matrix.

If $x \in [0,1]$ then the last category can be merged with the third one. First let us prove that a positive matrix $B \in \Omega_3$ cannot be a star. To prove this we require the following two lemmas.

Lemma 3.1. $B=(b_{ij}) \in \Omega_3$ *is not a star if there exists a matrix* $B+E$ *in* Ω_3 *where* $E \neq 0$ *such that*

$$\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(E_{ij}) + 2\,\mathrm{per}(E) \leqslant 0. \tag{3.1}$$

Proof. Using (1.3), a sufficient condition B not to be a star is that there exists $A(\neq B)$ in Ω_3 such that

$$\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(A_{ij}) - \mathrm{per}(B) - 2\,\mathrm{per}(A) \geqslant 0. \tag{3.2}$$

Taking $A = B + E$, (3.2) reduces to

$$\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(B_{ij} + E_{ij}) - \mathrm{per}(B) - 2\,\mathrm{per}(B + E) \geqslant 0. \tag{3.3}$$

Using formula (1.4) to expand $\mathrm{per}(B_{ij} + E_{ij})$ and $\mathrm{per}(B + E)$ and also making use of the relation

$$\sum_{i,j=1}^{3} b_{ij} S_1(E_{ij}, B_{ij}) = 2S_1(E, B)$$

and

$$\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(E_{ij}) = S_2(E, B)$$

condition (3.3) reduces to

$$\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(E_{ij}) + 2\,\mathrm{per}(E) \leqslant 0. \qquad \square$$

Lemma 3.2. *$B = (b_{ij}) \in \Omega_3$ is not a star if there exists a diagonal whose sum is d and minimum entry is ε such that $2d \leqslant 1 + 4\varepsilon$.*

Proof. Without loss of generality we can consider the diagonal (b_{11}, b_{22}, b_{33}) of B. Let $d = b_{11} + b_{22} + b_{33}$ and $\varepsilon = \min(b_{11}, b_{22}, b_{33})$.

Let

$$E = \begin{bmatrix} -\varepsilon & \varepsilon/2 & \varepsilon/2 \\ \varepsilon/2 & -\varepsilon & \varepsilon/2 \\ \varepsilon/2 & \varepsilon/2 & -\varepsilon \end{bmatrix}.$$

Obviously $\varepsilon > 0$ and $B + E \in \Omega_3$. By direct calculations

$$\begin{aligned}\sum_{i,j=1}^{3} b_{ij}\,\mathrm{per}(E_{ij}) + 2\,\mathrm{per}(E) &= (5\varepsilon^2/4)d - (\varepsilon^2/4)(3 - d) - 3\varepsilon^3 \\ &= (3\varepsilon^2/4)(2d - 1 - 4\varepsilon) \leqslant 0.\end{aligned}$$

Therefore, from Lemma 3.1, it follows that B is not a star. $\square$

Theorem 3.1. *A 3×3 positive doubly stochastic matrix cannot be a star.*

Proof. Let $B = (b_{ij})$ where $b_{ij} > 0$ for $i, j = 1, 2, 3$. Let d be the minimum diagonal sum of B. Without loss of generality assume $b_{11} + b_{22} + b_{33} = d$. Also we can assume that $\min(b_{11}, b_{22}, b_{33}) = b_{11}$. Since $b_{11} + b_{22} + b_{33}$ is the minimum diagonal sum, we have

$$b_{11} + b_{22} + b_{33} \leqslant b_{11} + b_{23} + b_{32}.$$

Therefore,

$$b_{22} + b_{33} \leqslant 1 - (b_{13} + b_{33}) + 1 - (b_{12} + b_{22}),$$

i.e., $2b_{22} + 2b_{33} - b_{11} \leqslant 1$.

Hence, $2b_{22} + 2b_{33} - 2b_{11} \leqslant 1 - b_{11} \leqslant 1$ implying that $2d \leqslant 1 + 4b_{11}$. Thus we find a diagonal satisfying condition of Lemma 3.2. Hence, B is not a star. □

Next, let us prove that the matrix B having precisely one or two or three zeros cannot be a star.

Theorem 3.2. *If the number of zeros in $B \in \Omega_3$ is exactly one or two or three, then B cannot be a star.*

Proof. If the number of zeros in $B = (b_{ij}) \in \Omega_3$ is one or two or three then obviously they will lie in different rows and different columns. Therefore, without loss of generality we can assume that $b_{11} = 0$, $b_{22} \geqslant 0$, $b_{33} \geqslant 0$ and $b_{ij} > 0$ for $i \neq j$.

Let

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & -\varepsilon \\ 0 & -\varepsilon & \varepsilon \end{bmatrix}.$$

For arbitrarily small $\varepsilon > 0$, $B + E \in \Omega_3$. For this E,

$$\sum_{i,j=1}^{3} b_{ij} \operatorname{per}(E_{ij}) + 2 \operatorname{per}(E) = 2b_{11}\varepsilon^2 = 0 \quad \text{since } b_{11} = 0.$$

Hence, from Lemma 3.1 it follows that B cannot be a star. □

We have proved in Section 2 that the matrix PBQ where $B = M(x, 1-x; 1-x, x)$ and P and Q are permutation matrices, is a star for $0 \leqslant x \leqslant 1$. Thus we arrive at the following main result of this section.

Theorem 3.3. *The only star in Ω_3 is $B = M(x, 1-x; 1-x, x)$, $0 \leqslant x \leqslant 1$ upto permutations of rows and columns.*

4. Concluding remarks

In [5] Wang defined that two matrices $A, B \in \Omega_n$ where $A \neq B$ are said to form a permanental pair if $\operatorname{per}(\alpha A + (1-\alpha)B) = \text{constant}$ for all $\alpha \in [0,1]$ in which case A and B are called permanental mates or simply mates of each other. Wang proved that matrices in Ω_2 and $n \times n$ permutation matrices have no mates. Later Brenner and Wang [6] established that $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $k \geqslant 1$, $1 \leqslant n_i \leqslant n$, $n_1 + \cdots + n_k = n$ has no mate where J_m denotes the $m \times m$ matrix all of whose entries are $1/m$. Gibson [7] found one more class of matrices, namely $M(x, 1-x; 1-x, x)$ $0 \leqslant x \leqslant 1$, having no mate. We deduce this result due to Gibson from our main result. For this we require the following theorem which connects the two concepts namely, mates and stars introduced by Wang.

Theorem 4.1. *If $B \in \Omega_n$ has a mate, then B is not a star.*

Proof. Let $A = (a_{ij}) \in \Omega_n$ be a mate of $B = (b_{ij}) \in \Omega_n$. Then $A \neq B$ and $\operatorname{per}(A) = \operatorname{per}(B)$. Since a necessary condition for A and B to be mates (refer [7]) is $S_1(B, A) = n \operatorname{per}(A)$, we have

$$\sum_{i,j=1}^{n} b_{ij} \operatorname{per}(A_{ij}) = n \operatorname{per}(A). \tag{4.1}$$

If B is a star then from the result of Brualdi and Newman quoted in (1.3) (with strict inequality since $A \neq B$),

$$\sum_{i,j=1}^{n} b_{ij} \operatorname{per}(A_{ij}) < \operatorname{per}(B) + (n-1) \operatorname{per}(A)$$

i.e., $n \operatorname{per}(A) < \operatorname{per}(B) + (n-1)\operatorname{per}(A)$

i.e., $\operatorname{per}(A) < \operatorname{per}(B)$ which is a contradiction.

Hence, we conclude that B is not a star. □

The above result can be stated equivalently as, if $B \in \Omega_n$ is a star then B cannot have a mate in Ω_n.

As a consequence of Theorems 2.1 and 4.1 we have the following result.

Corollary 4.1. *$M(x, 1-x; 1-x, x)$ for $0 \leqslant x \leqslant 1$, upto permutations of rows and columns cannot have a mate.*

We conclude this paper by proposing the following conjectures.

Conjecture 4.1. The direct sum of two stars is also a star.

Conjecture 4.2. The only stars in Ω_n are the direct sum of 2×2 doubly stochastic matrices and identity matrices upto permutations of rows and columns.

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